1 a) Let $h(k) = 0 \ \forall k \in \mathbb{N}$. Since T is a linear operator, $T(0) = 0$ and thus, $h(k+1) = T(h(k)) \ \forall k \in \mathbb{N}$. Hence, $h \in S$ and thus S is nonempty.

Next, let $f, g \in S$. Since T is linear, $T(f(k) + g(k))$
$= T(f(k)) + T(g(k)) = f(k+1) + g(k+1), \ \forall k \in \mathbb{N}$.
Therefore, $f + g \in S$.

Finally, let $f \in S$ and $a \in \mathbb{R}$. again, using linearity of T, $T(af(k)) = aT(f(k)) = af(k+1) \ \forall k \in \mathbb{N}$.
Therefore, $af \in S$.

We conclude that S is a subspace.

b) Let $x \in \mathbb{R}^n$.
Define $f : \mathbb{N} \to \mathbb{R}^n$ by $f(k) = T^{k-1}(x)$ for $k \in \mathbb{N}$.

Then $f(k+1) = T^k(x) = T(T^{k-1}(x)) = T(f(k))$ for all $k \in \mathbb{N}$.
So $f \in S$.

c) Let $\{x_1, x_2, \dots, x_n\}$ be a basis of $\mathbb{R}^n$.
We claim that $\{f_1, f_2, \dots, f_n\}$, defined by

$$f_i(k) := T^{k-1}(x_i) \qquad i = 1, 2, \dots, n, \qquad k \in \mathbb{N}$$

is a basis of S.
By b), $f_i \in S$ for all $i = 1, 2, \dots, n$.

Suppose $\sum_{i=1}^{n} a_i f_i = 0$ for $a_i \in \mathbb{R}$

Then $\sum_{i=1}^{n} a_i f_i(1) = 0 \implies \sum_{i=1}^{n} a_i x_i = 0$

$\implies a_1 = a_2 = \dots = a_n = 0$ since $\{x_1, \dots, x_n\}$ is a basis of $\mathbb{R}^n$.
$\implies f_1, f_2, \dots, f_n$ are linearly independent.

Finally, let $g \in S$. Since $g(k+1) = T(g(k))$ for all $k \in \mathbb{N}$, $g(k) = T^{k-1}(g(1)) \ \forall \ k \in \mathbb{N}$.
Since $\{x_1, x_2, \dots, x_n\}$ is a basis of $\mathbb{R}^n$, we can express

$$g(1) = \sum_{i=1}^{n} a_i x_i$$

where $a_i \in \mathbb{R}$ for $i = 1, 2, \dots, n$.

But then $g(k) = T^{k-1}\left(\sum_{i=1}^{n} a_i x_i\right)$

$= \sum_{i=1}^{n} a_i T^{k-1}(x_i)$ (since T^{k-1} is linear by linearity of T).

$= \sum_{i=1}^{n} a_i f_i(k)$

So $g = \sum_{i=1}^{n} a_i f_i \implies S = \operatorname{span}(f_1, f_2, \dots, f_n)$.

We conclude that $\{f_1, f_2, \dots, f_n\}$ is a basis of S.

4 a) Let

$$G := \begin{pmatrix} k(x_1, x_1) & \cdots & k(x_1, x_n) \\ \vdots & & \vdots \\ k(x_n, x_1) & \cdots & k(x_n, x_n) \end{pmatrix}$$

be the Gram matrix. By definition, G is symmetric and positive semidefinite.

Since G is symmetric, it has real eigenvalues.

Let $\lambda \in \mathbb{R}$ be an eigenvalue of G with eigenvector $v \in \mathbb{R}^n \setminus \{0\}$.

Then $Gv = \lambda v \implies v^T G v = \lambda v^T v = \lambda \|v\|^2$

So $\lambda = \frac{v^T G v}{\|v\|^2} \geq 0$ since $v^T G v \geq 0$ by positive semidefiniteness of G and $\|v\| > 0$ since $v \neq 0$.

b) Take $k(x,y) = \langle f(x), f(y) \rangle$.
Let $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in \mathcal{X}$.
The Gram matrix is now:

$$G = \begin{pmatrix} \langle f(x_1), f(x_1) \rangle & \cdots & \langle f(x_n), f(x_1) \rangle \\ \vdots & & \vdots \\ \langle f(x_1), f(x_n) \rangle & \cdots & \langle f(x_n), f(x_n) \rangle \end{pmatrix}.$$

Clearly, since $\langle f(x_i), f(x_j) \rangle = \langle f(x_j), f(x_i) \rangle$ for all $i,j = 1,2,\dots,n$, the matrix G is symmetric.
Now, let $v \in \mathbb{R}^n$.
Then

$$v^T G v = v^T \begin{pmatrix} \sum_{i=1}^n \langle f(x_i), f(x_1) \rangle v_i \\ \vdots \\ \sum_{i=1}^n \langle f(x_i), f(x_n) \rangle v_i \end{pmatrix}$$

$$= \sum_{j=1}^n v_j \sum_{i=1}^n \langle f(x_i), f(x_j) \rangle v_i$$

$$= \sum_{j=1}^n v_j \left\langle \sum_{i=1}^n v_i f(x_i), f(x_j) \right\rangle \quad \text{(using linearity in first arg.)}$$

$$= \sum_{j=1}^n v_j \left\langle f(x_j), \sum_{i=1}^n v_i f(x_i) \right\rangle \quad \text{(using symmetry)}$$

$$= \left\langle \sum_{j=1}^n v_j f(x_j), \sum_{i=1}^n v_i f(x_i) \right\rangle$$

$\geq 0 \qquad$ (using nonnegativity of $\langle \cdot, \cdot \rangle$).

So G is symmetric positive semidefinite $\Rightarrow$ k is positive semidefinite.

c) Let $k(x,y) = (xy+1)^2$.
Note that $k(0,0) = 1$, $k(1,0) = 1$, $k(2,0) = 1$, $k(3,0) = 1$, $k(1,1) = 4$, $k(2,1) = 9$, $k(3,1) = 16$, $k(2,2) = 25$, $k(3,2) = 49$, $k(3,3) = 100$. Moreover, $k(x,y) = k(y,x)$ $\forall x,y \in \mathbb{R}$.
So the given matrix is equal to the Gram matrix

$$G = \begin{bmatrix} k(x_1,x_1) & \cdots & k(x_1,x_4) \\ \vdots & & \vdots \\ k(x_4,x_1) & \cdots & k(x_4,x_4) \end{bmatrix}$$

with $x_i = i-1$ for $i = 1,2,3,4$.

We can write $k(x,y) = 1 + 2xy + x^2y^2$

$$= \begin{pmatrix} 1 & \sqrt{2}\,x & x^2 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2}\,y \\ y^2 \end{pmatrix}.$$

So for $\mathcal{V} = \mathbb{R}^3$, $\mathcal{X} = \mathbb{R}$, $f : \mathcal{X} \to \mathcal{V}$
given by $f(x) = \begin{pmatrix} 1 \\ \sqrt{2}x \\ x^2 \end{pmatrix}$,

we get $k(x,y) = \langle f(x), f(y) \rangle$, where $\langle \cdot, \cdot \rangle$
is the standard inner product (dot product) on $\mathbb{R}^3$.

By (b), k is positive semidefinite. So by
definition, G is symmetric and positive semidefinite.

Ex. 2. $\langle \cdot, \cdot \rangle_V$, $\langle \cdot, \cdot \rangle_W$ inner prod.

$T: V \to W$.

unitary if $\langle v_1, v_2 \rangle_V = \langle T(v_1), T(v_2) \rangle_W$.

a) $V = \mathbb{C}^2$, $W = \mathbb{C}^3$ (standard Herm. inner prod.)

show that $T: V \to W$ given by $\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ w \\ 0 \end{pmatrix}$ is unitary but not surj.

Let $v_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$ $v_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$ $T(v_1) = \begin{pmatrix} z_1 \\ w_1 \\ 0 \end{pmatrix}$.

$T(v_2) = \begin{pmatrix} z_2 \\ w_2 \\ 0 \end{pmatrix}$

$\langle v_1, v_2 \rangle_V = \left\langle \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \end{pmatrix} \right\rangle = z_1 \overline{z_2} + w_1 \overline{w_2}$

$\langle T(v_1), T(v_2) \rangle_W = \left\langle \begin{pmatrix} z_1 \\ w_1 \\ 0 \end{pmatrix}, \begin{pmatrix} z_2 \\ w_2 \\ 0 \end{pmatrix} \right\rangle =$

$= z_1 \overline{z_2} + w_1 \overline{w_2} + 0 = z_1 \overline{z} + w_1 \overline{w_2}$

$\Rightarrow \langle v_1, v_2 \rangle_V = \langle T(v_1), T(v_2) \rangle_W$.

~~(b)~~

$\Rightarrow T$ unitary

Consider $\begin{pmatrix}0\\0\\1\end{pmatrix} \in \mathbb{C}^3$. There is no vector $\begin{pmatrix}z\\w\end{pmatrix} \in \mathbb{C}^2$ s.t.

$$T\left(\begin{pmatrix}z\\w\end{pmatrix}\right) = \begin{pmatrix}0\\0\\1\end{pmatrix}$$, because the third component of the image under T is always 0, $\forall$ input vector.

$\Rightarrow T$ not surjective

b) Prove that every unitary T is injective

Suppose $\exists\ v_1, v_2 \in V$ s.t. $T(v_1) = T(v_2)$

Then by def. of unitary transf:

$$\left.\begin{array}{l} \langle v_1, v_2 \rangle_V = \langle T(v_1), T(v_2) \rangle_W \\ \text{since } T(v_1) = T(v_2) \end{array}\right\} \Rightarrow$$

$\langle v_1, v_2 \rangle_V = \langle T(v_1), T(v_1) \rangle_W = 1$

$\Leftrightarrow \quad v_1 = v_2 \quad \Rightarrow T$ is injective

c). Suppose $\dim V = \dim W < \infty$.

" $T: V \to W$ unitary.

Prove T is bijective.

b) $\Rightarrow T$ injective

$\hookrightarrow$ injective linear transf. between finite dim. vector spaces of the same dime $\Rightarrow T$ surj.

$\Rightarrow T$ bijective

d) show T^{-1} is the adjoint of T

T bijective $\Rightarrow T^{-1} \exists$

The adjoint of T, denoted T^* is defined as the unique lin. transf satisfying $\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V$

$\forall v \in V$
$w \in W$

e T is bijective $\Rightarrow T^{-1}$ $\exists$ and satisfies the above property $\Rightarrow$

$$\Rightarrow T^{-1} = T^*$$

$$\Big(\langle Tv, w \rangle_W = \langle Tv, TT^{-1}w \rangle_W =$$

$$= \langle v, T^{-1}w \rangle_V$$

T^*.